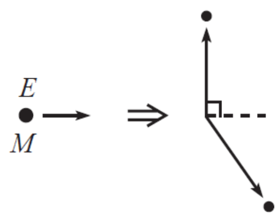
Physics IV
ISI B.Math
Final Exam : May 6, 2022

Total Marks: 50
Time : 3 hours
Answer all questions



1.(**Marks: 5 + 5 = 10**)

(a) A particle of mass M and energy E decays into two identical particles. In the lab frame, one is emitted at a 90 degree angle as shown in the figure. What are the energies of the created particles in the lab frame ?

(b) Let V be a four-vector. Show that if V is future pointing timelike, there exists a coordinate system in which it has components $(a, 0, 0, 0)$ where $a = \sqrt{V^\mu V_\mu}$

2. (**Marks : 1 + 2 + 2 + 5 = 10**)

A one dimensional harmonic oscillator of mass m has potential energy $V(x) = \frac{1}{2}m\omega^2 x^2$. Consider the operators $a = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x + ip)$ and $a^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x - ip)$

(a) Do a and $a^\dagger$ correspond to physically measurable observables ? Justify your answer.

(b) Express the Hamiltonian $\hat{H}$ in terms of the number operator $N = a^\dagger a$ Find the eigenvalues of the energy in terms of the eigenvalues n of the number operator where $n = 0, 1, 2 \cdots$

(c) Show that if $|n>$ is an eigenvector of the number operator N with eigenvalue n, then $a^\dagger|n>$ is an eigenvector of N with eigenvalue $n + 1$

(d) Find the expectation value of the potential energy $< V >$ in the state $|n>$. Show that the ground state $|0>$ is a minimum uncertainty state

3. (**Marks : 5 + 2 + 3 = 10**)

A particle of mass m is moving in an infinite potential well of width a under the potential $V(x) = 0$ for $0 \leq x \leq a$ and $V(x) = \infty$ otherwise.

(a) Find the possible values of the energies E_n of the stationary states and the corresponding wave functions $\psi_n(x)$(solutions of the time independent Schrodinger equation)

(b) If the particle starts out in starts out in the left half of the well, and is at $(t = 0)$ equally likely to be found at any point in that region, what is the initial wave function $\Psi(x, 0)$? (Assume it is real)

(c) With the particle in the initial state described in part (b) , what is the probability that a measurement of energy would yield the value $\frac{\pi^2\hbar^2}{2ma^2}$?

4. (**Marks : 3 + 4 + 3 = 10**)

(a) Show that the energy E must exceed the minimum value of $V(x)$ for every normalizable solution to the time independent Schrödinger equation in one dimension.

(b) Let $P_{ab}(t)$ be the probability of finding a particle of mass m in the range $(a < x < b)$ at time t. Show that

$$\frac{dP_{ab}}{dt} = J(a, t) - J(b, t)$$

where $J(x, t) = \frac{i\hbar}{2m}\left(\Psi(x, t)\frac{\partial\Psi^*(x,t)}{\partial x} - \Psi^*(x, t)\frac{\partial\Psi(x,t)}{\partial x}\right)$ and $\Psi(x, t)$ is the wave function of the particle

(c) If $V(x)$ is an even function of x, then the solutions $\psi(x)$ of the one-dimensional time-independent Schrödinger equation can always be taken to be either even or odd.

5. (**Marks : 2 + 4 + 4 =10**)

(a) A free particle of mass m moves in one dimension. At time $t = 0$, the normalized wave function of the particle is

$$\psi(x, 0) = (2\pi\sigma_x^2)^{-\frac{1}{4}} \exp\left(-\frac{x^2}{4\sigma_x^2}\right)$$

(i) Is the particle in a state of definite energy ? Explain

(ii) Find the spread in momentum $\sigma_p = \sqrt{< p^2 > - < p >^2}$ of the particle in this state. This particle is said to be in a state of minimum uncertainty. Justify the statement. Will this property continue to hold for $t > 0$?

(b)Consider an observable represented by an operator $\hat{A}$ which does not explicitly depend on time and whose commutator with the Hamiltonian $\hat{H}$ is the constant c, $[\hat{H}, \hat{A}] = c$. Find $< \hat{A} >$ at $t > 0$, given that the system is in a normalized eigenstate of $\hat{A}$ at $t = 0$, corresponding to the eigenvalue a.